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COMMENT

Dual properties of the percolation lattices

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Abstract. We attempt to generalise the dual relation $p_c(\mathcal{L}, d) + p_c(\mathcal{L}_D, d) = 1$, obtained for a dual pair of planar lattices denoted by $L(\mathcal{L}, d)$ and $L(\mathcal{L}_D, d)$. Here, \mathcal{L} and \mathcal{L}_D are the coordination numbers of these lattices, d their dimension, and p_c is the bond percolation threshold.

Percolation theory shows potential applications to a variety of fields possessing many striking parallels with thermal phenomena. Despite the apparent simplicity of percolation, few exact results have been reported for the most interesting properties: the percolation threshold and the critical exponents (for reviews, see Kirkpatrick 1979 and Stauffer 1985). In fact, the percolation threshold is well established exactly for only some cases: the square lattice bond, triangular and Kagomé lattice sites, (Sykes and Essam 1964, Kesten 1980, Wierman 1981) and the Bethe lattices (Fisher and Essam 1961).

In this context, some approximate formulae are also available for the bond percolation threshold $p_c(\mathcal{L}, d)$ where \mathcal{L} and d are respectively the coordination number and the dimension of the lattice denoted by $L(\mathcal{L}, d)$. In particular, Vyssotsky *et al* (1961) showed that:

$$p_c(\mathcal{L}, d) = \frac{d}{(d-1)\mathcal{L}} \tag{1}$$

This empirical relation appears to be a good approximation for $2 \leq d \leq 3$. A single-bond effective medium (Kirkpatrick 1973) yields:

$$p_c(\mathcal{L}, d) = \frac{2}{\mathcal{L}} \tag{2}$$

which leads to the trivial value of the linear chain ($d = 1$), and is equivalent to (1) for $d = 2$. Sahimi *et al* (1983) also observed that for $d \geq 3$, $p_c \approx G_0$ where G_0 denotes a Bravais lattice Green function. Furthermore, exact expansions for the mean cluster numbers and the mean cluster sizes (Gaunt *et al* 1976, Gaunt and Brack 1984) yield satisfactory approximations for the simple cubic lattices $L(\mathcal{L}_0 = 2d, d)$ in $d \geq 3$.

For the particular case of the planar lattices ($d = 2$), a dual relation has been reported previously (Sykes and Essam 1964):

$$p_c(\mathcal{L}, d) + p_c(\mathcal{L}_D, d) \geq 1 \tag{3}$$

where \mathcal{Z}_D is the coordination number of the dual lattice $L(\mathcal{Z}_D, d)$. However, to date, only strict equality has been verified. It is interesting to generalise the dual relation for $1 \leq d \leq 3$. In this sense, we consider a linear variation of the bond percolation threshold p_c on the 'reduced length of coordination $Y = d/\mathcal{Z}$ ' for a given dimension d :

$$p_c(Y) = AY + B \tag{4}$$

where A and B are dimensional constants.

If we assume that the bond percolation threshold of the simple hypercubic lattice $L(\mathcal{Z}_0 = 2d, d)$ is known, (4) takes the form:

$$p_c(\mathcal{Z}, d) = \frac{d}{a_d \mathcal{Z}} + \frac{B(\mathcal{Z} - 2d)}{\mathcal{Z}} \tag{5}$$

where $a_d = 1/2p_c(\mathcal{Z}_0, d)$.

The literature results (Stauffer 1985): $p_c = 1$ for the linear chain, $p_c = \frac{1}{2}$ for the square lattice and $p_c = 0.2495$ for the simple cubic lattice, show that $p_c(\mathcal{Z}_0, d) = 2^{1-d}$ seems to be a good approximation for the simple hypercubic lattice for $d \leq 3$.

Then (5) may be written as:

$$p_c(\mathcal{Z}, d) = \begin{cases} \frac{d}{(d-1)\mathcal{Z}} + f(\mathcal{Z}, d, B) & 2 \leq d \leq 3 & (6a) \\ \frac{2}{\mathcal{Z}} + f(\mathcal{Z}, d, B) & d \leq 2 & (6b) \\ \frac{d}{2^{d-2}\mathcal{Z}} + f(\mathcal{Z}, d, B) & \text{any } d & (6c) \end{cases}$$

with $f(\mathcal{Z}, d, B) = B(\mathcal{Z} - 2d)/\mathcal{Z}$. When \mathcal{Z} approaches \mathcal{Z}_0 , we find respectively (1) and (2).

To obtain a dual relation similar to (3), we consider the duality transformation. The dual lattice $L(\mathcal{Z}_D, d)$ may be obtained from the lattice $L(\mathcal{Z}, d)$ by substituting each loop of $L(\mathcal{Z}, d)$ with a site. Therefore, two sites are joined by a bond in $L(\mathcal{Z}_D, d)$ if their corresponding loops present a common face (see figure 1). A given lattice is self-dual if it is identical to its dual lattice.

Denoting the bond length by t and the angle between two nearest-neighbouring faces by $\theta(\mathcal{Z}, d)$:

$$t_D = ttg\left(\frac{\theta(\mathcal{Z}, d)}{2}\right) \tag{7a}$$

and reciprocally,

$$t = t_D tg\left(\frac{\theta(\mathcal{Z}_D, d)}{2}\right). \tag{7b}$$

Combining (7a) and (7b), it follows that:

$$tg\left(\frac{\theta(\mathcal{Z}, d)}{2}\right)tg\left(\frac{\theta(\mathcal{Z}_D, d)}{2}\right) = 1 \tag{8}$$

whose solution is given by:

$$\theta(\mathcal{Z}, d) + \theta(\mathcal{Z}_D, d) = \pi \tag{9}$$

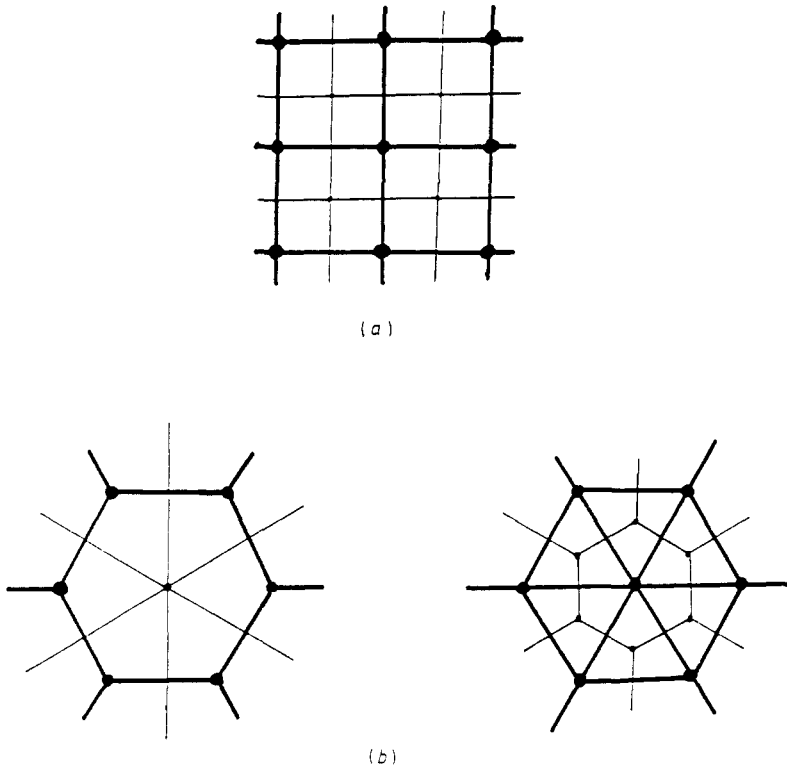


Figure 1. The dual lattices (denoted by thin lines): (a) the square lattice is self dual; (b) the triangular and honeycomb lattices form a dual pair.

Relation (9) may be written as:

$$\theta(\mathcal{L}, d) + \theta(\mathcal{L}_D, d) = 2\theta(\mathcal{L}_0, d). \tag{10}$$

Setting $\theta(\mathcal{L}, d) = d\pi/\mathcal{L}$, (10) yields:

$$Y + Y_D = 2Y_0 = 1. \tag{11}$$

Relation (11) can give us directly the dual lattice of a given lattice for any dimension d .

Using the linear character of $p_c(\mathcal{L}, d)$ and (11), we obtain

$$p_c(\mathcal{L}, d) + p_c(\mathcal{L}_D, d) = 2p_c(\mathcal{L}_0, d). \tag{12}$$

Inserting the approximation $p_c(\mathcal{L}_0, d) = 2^{1-d}$, it follows that

$$p_c(\mathcal{L}, d) + p_c(\mathcal{L}_D, d) = 2^{2-d}. \tag{13}$$

Such a relation is exactly established for the linear chain and for the planar lattices.

For $d = 3$, Monte Carlo simulations (Stauffer and Zabolitsky 1986) give $p_c \approx 0.2495$ for the simple cubic lattice, $p_c \approx 0.388$ for the diamond lattice and $p_c \approx 0.119$ for the face-centred cubic. From (11), it appears that the simple cubic is self-dual while the diamond and the face-centred lattices form a dual pair. Under these considerations, (13) seems to be a satisfactory candidate for $d \leq 3$.

For $d \geq 4$, very little has been carried out, i.e. only results on hypercubic lattices have been reported (Gaunt *et al* 1984). Therefore, it should be difficult to investigate and draw any conclusions about relation (12).

In summary, we have presented here two dual realtions based on the coordination numbers and on the bond percolation thresholds of a dual pair of lattices. It is shown that the self-duality will concern all the simple hypercubic lattices and not just the planar lattices.

The linear variation of the bond percolation threshold on Y seems to be only appreciable for the lowest dimensions $d \leq 3$ as analysed by Gaunt and Brack (1984).

References

- Fisher M E and Essam J W 1961 *J. Math. Phys.* **2** 609
Gaunt D S, Sykes M F and Ruskin H 1976 *J. Phys. A: Math. Gen.* **9** 1899
Gaunt D S and Brack R 1984 *J. Phys. A: Math. Gen.* **17** L61
Kesten H 1980 *Commun. Math. Phys.* **74** 41
Kirkpatrick S 1979 *III-Condensed Matter* ed R Balian *et al* (Amsterdam: North-Holland) p 324
Kirkpatrick S 1973 *Rev. Mod. Phys.* **45** 574
Sahimi M, Hughes D B, Scriven L E and Davis H T 1983 *J. Phys. A: Math. Gen.* **16** L67
Stauffer D 1985 (ed) *Introduction to Percolation Theory* (London: Taylor and Francis) p 1
Stauffer D and Zabolitzky J G 1986 *J. Phys. A: Math. Gen.* **19** 3405
Sykes M F and Essam J W 1964 *J. Math. Phys.* **5** 1117
Vyssotostky V A, Gordon S B, Frish H L and Hammersley J M 1961 *Phys. Rev.* **123** 1566
Wierman J C 1981 *Adv. Appl. Prob.* **13** 298